

Nonlinear wave interaction and spin models in the MHD regime

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Here we consider the influence on the electron spin in the MHD regime. Recently developed models which include spin-velocity correlations are taken as a starting point. A theoretical argument is presented, suggesting that in the MHD regime a single fluid electron model with spin correlations is equivalent to a model with spin-up and spin-down electrons constituting different fluids, but where the spin-velocity correlations are omitted. Three wave interaction of 2 shear Alfvén waves and a compressional Alfvén wave is then taken as a model problem to evaluate the asserted equivalence. The theoretical argument turns out to be supported, as the predictions of the two models agree completely. Furthermore, the three wave coupling coefficients obey the Manley-Rowe relations, which give further support to the soundness of the models and the validity of the assumptions made in the derivation. Finally we point out that the proposed two-fluid model can be incorporated in standard Particle-In-Cell schemes with only minor modifications.

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I. INTRODUCTION

Considerable interest has recently been devoted to the study of quantum plasmas, see e.g. Refs [1–8]. Much of the research has been motivated by applications to e.g. quantum wells [9], plasmonics [10], spintronics [11], astrophysics [12] and ultra-cold plasmas [13]. Two of the most basic and much studied quantum effects are those of the Fermi pressure and the particle dispersive effects (directly associated with the Bohm De Broglie potential), see e.g. Refs. [1–8]. Other studies [3, 14–21] focus on the electron spin properties that result in the magnetic dipole force and a magnetization current, in addition to some more complex aspects of the spin dynamics. Although most quantum effects has a tendency to be more important in plasmas of high density and low temperature, the regimes of relevance differ to some extent for the various quantum effects, see Ref. [19] for a discussion of this issue. A consequence is that it is possible to focus on certain of the quantum effects and ignore the others. In this paper we will make use of this fact and concentrate on the physics associated with the spin coupling in the Pauli Hamiltonian. Our starting point is a recently presented spin-fluid model [16], derived from kinetic theory [15], which in addition to the most basic spin precession dynamics includes effects of spin-velocity correlations. Evaluating this model in the MHD regime, we make a conjecture based on certain teoretical arguments; That there are two equivalent ways to model spin-MHD dynamics, either by a one-fluid model including spin-velocity correlations, or by a two-fluid model without spin-velocity correlations. In the latter case the spin-up and spin-down states relative to the magnetic field are regarded as different fluids [14]. The conjectured equivalence of these models in the MHD regime is tested by considering a specific problem of three-wave interaction. For this purpose we calculate the coupling coefficients between two shear Alfvén waves and one compressional Alfvén wave (fast magnetosonic wave) in a magnetized plasma. The coupling coefficients turns indeed out to be identical in the two cases, and the coefficients are also seen to obey the Manley-Rowe relations, which give further support to the soundness of the models used. The applicability of the two-fluid model *without spin-velocity correlations in the MHD-regime*, which is strongly supported by our findings, is a very useful result. The reason is that as this model can be easily adopted into standard Particle-In-Cell schemes with only small modifications, as will be discussed in the final section.

II. MODEL EQUATIONS

Starting from a scalar kinetic equation for a spin-1/2 particle [15], spin fluid equations can be derived [16]. These are given by the continuity equation

$$\partial_t n^{(s)} + \nabla \cdot (n^{(s)} \mathbf{v}^{(s)}) = 0 \quad (1)$$

and the fluid momentum equation

$$m^{(s)} \frac{Dv_i^{(s)}}{Dt} = q^{(s)} \left(E_i + \varepsilon_{ijk} v_j^{(s)} B_k \right) + \mu^{(s)} S_j^{(s)} \frac{\partial B_j}{\partial x_i} - \frac{1}{n^{(s)}} \frac{\partial P_{ij}^{(s)}}{\partial x_j}, \quad (2)$$

where the superscript $s = e, i$ denotes the species (electrons or ions), $D/Dt \equiv \partial_t + \mathbf{v}^{(s)} \cdot \nabla$, and summation over repeated indices $i, j, k = x, y, z$ is implied. Here $m^{(s)}$ is the mass, $q^{(s)}$ is the charge, $\mu^{(s)}$ the magnetic dipole moment,

$n^{(s)}$ is the number density, and $\mathbf{v}^{(s)}$ is the fluid velocity of species s . Furthermore, \mathbf{S} is the spin vector normalized to unity, P_{ij} is the pressure tensor, and ε_{ijk} is the Levi-Civita symbol. Since the ions normally have a much smaller magnetic moment than the electrons [22], the spin contribution due to the ions can be neglected compared to the electron contribution, i.e. we may let $\mu^{(i)} \approx 0$. We have here neglected contributions to the force from the Fermi pressure and particle dispersive effects (the so called Bohm de Broglie potential), see [8]. For a discussion of the parameter regime of importance of various quantum effects, see e.g. Refs. [19, 23]. In this approximation, the pressure moment satisfies the evolution equation

$$\begin{aligned} \frac{DP_{ij}^{(s)}}{Dt} = & -P_{ik}^{(s)} \frac{\partial v_j^{(s)}}{\partial x_k} - P_{jk}^{(s)} \frac{\partial v_i^{(s)}}{\partial x_k} - P_{ij}^{(s)} \frac{\partial v_k^{(s)}}{\partial x_k} + \frac{q^{(s)}}{m^{(s)}} \varepsilon_{imn} P_{jm}^{(s)} B_n + \frac{q^{(s)}}{m^{(s)}} \varepsilon_{jmn} P_{im}^{(s)} B_n \\ & + \frac{\mu^{(s)}}{m^{(s)}} \Sigma_{ik} \frac{\partial B_k}{\partial x_j} + \frac{\mu^{(s)}}{m^{(s)}} \Sigma_{jk} \frac{\partial B_k}{\partial x_i}, \end{aligned} \quad (3)$$

where again the last two terms can be dropped for the ion species.

Furthermore, to describe the spin dynamics we need the electron spin evolution equation, which is given by

$$\frac{DS_i^{(e)}}{Dt} = \frac{2\mu^{(e)}}{\hbar} \varepsilon_{ijk} S_j^{(e)} B_k - \frac{1}{m^{(e)} n^{(e)}} \frac{\partial \Sigma_{ij}^{(e)}}{\partial x_j} \quad (4)$$

where $\Sigma_{ij}^{(e)}$ is the spin-velocity correlation tensor. Finally the evolution of the spin-velocity moment is described by

$$\begin{aligned} \frac{D\Sigma_{ij}^{(e)}}{Dt} = & -\Sigma_{ij}^{(e)} \frac{\partial v_k^{(e)}}{\partial x_k} - \Sigma_{ik}^{(e)} \frac{\partial v_j^{(e)}}{\partial x_k} - P_{jk}^{(e)} \frac{\partial S_i^{(e)}}{\partial x_k} \\ & + \frac{q^{(e)}}{m^{(e)}} \varepsilon_{jkl} \Sigma_{ik}^{(e)} B_l + \frac{2\mu^{(e)}}{\hbar} \varepsilon_{ikl} \Sigma_{kj}^{(e)} B_l + \mu^{(e)} n^{(e)} \frac{\partial B_i}{\partial x_j} - \mu^{(e)} n^{(e)} S_i^{(e)} S_k^{(e)} \frac{\partial B_k}{\partial x_j} \end{aligned} \quad (5)$$

In Eq. (3) we have neglected the heat flux tensor Q_{ijk} to obtain a closed set of equations. Similarly we have neglected the higher order tensor Λ_{ijk} in the evolution equation for the spin-velocity tensor Eq. (5). The validity of the truncation has been investigated in Refs. [15, 24], and the truncation seem to be an accurate approximation in the low-temperature limit. The equations above together with Maxwell's equations constitute a closed system, where a magnetization current density $\mathbf{j}_M = \nabla \times \mathbf{M}$, due to the spin, should be added to the free current density, and where naturally all species contribute in the latter term. The set of equations (1)-(5) has been studied by Refs. [16, 24], but without inclusion of the ion dynamics. The aim of the current paper is to apply the above set of equations to the MHD regime where the ion dynamics is essential, at the same time as making a careful evaluation of the electron spin magnetization.

A. MHD-limit

As concluded in the previous section, we will primarily consider wave dynamics in the MHD regime where the frequencies are smaller than the ion-cyclotron frequency and the wavelengths are longer than the Larmor radius. Furthermore, we will consider the low-temperature (i.e. low-beta) limit, where the pressure terms are dropped. Under these assumptions, the system will be described by the magnetohydrodynamic equation [20]

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla \left(\frac{B^2}{2\mu_0} - \mathbf{M} \cdot \mathbf{B} \right) + (\mathbf{B} \cdot \nabla) \mathbf{M} - \nabla P_e \quad (6)$$

together with the equations

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}). \quad (7)$$

and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (8)$$

Here we have neglected the electron contribution to the fluid mass density by setting $\rho \approx m^{(i)} n^{(i)}$, and the fluid velocity can be written as $\mathbf{u} = (n^{(e)} m^{(e)} \mathbf{v}^{(e)} + n^{(i)} m^{(i)} \mathbf{v}^{(i)}) / \rho \approx \mathbf{v}^{(i)}$. We have neglected the magnetic moment of

the ions such that the magnetization $\mathbf{M} = \mu^{(e)} n^{(e)} \mathbf{S}^{(e)}$ is purely due to the electron spin. The derivation of Eq. (6) made in Ref. [20] was done starting from a somewhat less elaborate set of equations, not including the spin-velocity correlations. However, the derivation does not involve the Eqs. (4)-(5) describing the spin dynamics, and hence we may adopt this result within the current model.

Without the magnetization \mathbf{M} we obtain standard ideal MHD equations, and thus (6), (7) and (8) constitute a closed system. The magnetization can be easily included with the spin determined by Eqs. (4) together with (5), where the terms containing derivatives of the velocity turns out to be negligible. The aim is to solve for the magnetization in terms of the magnetic field, in which case Eqs. (6), (7) and (8) are sufficient to produce a closed spin-MHD theory. This can be achieved in two different ways; either by considering the electrons in spin up and spin down states relative the magnetic field as two separate fluids, or by treating them as a single fluid with a macroscopic spin that is proportional to the difference in population density of the two spin states. We will now go on to discuss this in further detail.

B. One-fluid vs two-fluid

In this sub-section we will discuss how to determine the magnetization, in order to use Eqs. (6), (7) and (8). To find the magnetization we first need to solve (4) to determine the spin. The first term of the right hand side of Eq. (4) is the basic spin precession. If the spin-velocity correlations in (4) can be neglected, the solutions for \mathbf{S} are particularly simple in the MHD regime. This is because the left hand side term of Eq. (4) is smaller than the spin precession term by a factor of the order $O(\omega^{(\text{ch})}/\omega_{cg}^{(\text{ch})})$, where $\omega^{(\text{ch})} \sim \partial_t$ is a characteristic frequency scale of the problem, and $\omega_{cg}^{(\text{ch})} \sim 2\mu_e B/\hbar$ is the characteristic spin precession frequency (which is close to the characteristic cyclotron frequency $\omega_c^{(\text{ch})} \sim qB/m$). Assuming that spin-velocity correlations can be omitted, we note that spin evolution equation in the MHD regime reduces to

$$\varepsilon_{ijk} S_j^{(e)} B_k = 0. \quad (9)$$

This has two solutions, where \mathbf{S} is either parallel or antiparallel to \mathbf{B} , that is $S_i = \pm b_i$, where $b_i = B_i/B$ is a unit vector in the direction of \mathbf{B} . A comparatively general way to deal with spins obeying $S_i = \pm b_i$ is to consider a two-fluid model of electrons, where for one of the species the electron spin state is parallel to \mathbf{B} , and for the other species antiparallel. Eq. (9) then implies that these spin states are conserved. However, as seen from the above discussion this is only an adequate approximation if the spin-velocity correlations give a small contribution in Eq. (4). Thus our next step is to outline the solutions of Eq. (5) using MHD approximations, in order to determine the contribution from $\Sigma_{ij}^{(e)}$ in (4). Firstly we note that the three first terms in (5) are at most of order $\omega^{(\text{ch})}\Sigma_{ij}$, whereas the fifth and sixth are of order $\omega_c^{(\text{ch})}\Sigma_{ij} \sim \omega_{cg}^{(\text{ch})}\Sigma_{ij}$. Thus neglecting the three first terms we can formally write Eq. (5) on the form $\overleftrightarrow{\mathcal{O}} \cdot \vec{\Sigma} = \vec{\sigma}$, where $\overleftrightarrow{\mathcal{O}}$ is a 9×9 -matrix where all coefficients are $\pm\omega_{c\alpha}$ or $\pm\omega_{cg\alpha}$. Here $\vec{\Sigma}$ is a 9-component vector containing all elements of Σ_{ij} , and $\vec{\sigma}$ is a 9-component vector containing the source terms, i.e. $P_{jk}(\partial S_i/\partial x_k)$, $\mu n(\partial B_i/\partial x_j)$ and $\mu n S_i S_k(\partial B_k/\partial x_j)$, $\omega_{c\alpha} = qB_\alpha/m$ and $\omega_{cg\alpha} = 2\mu_e B_\alpha/\hbar$ with $\alpha = x, y, z$. Note that the full field strength is used and not the linearized field when defining $\omega_{c\alpha}$ and $\omega_{cg\alpha}$. Since $\overleftrightarrow{\mathcal{O}}$ contains no operators, we can do a simple matrix inversion to find $\vec{\Sigma} = \overleftrightarrow{\mathcal{O}}^{-1} \vec{\sigma}$. This turns out to be sufficient to determine all components of Σ_{ij} , except for a component directed as $\mathbf{b} \otimes \mathbf{b}$. Thus this approximation scheme allows us to compute Σ_{ij} apart from a contribution that can be expressed as $\Sigma_{ij} = \Phi b_i b_j$, where Φ is a scalar field. Using the solution $\vec{\Sigma} = \overleftrightarrow{\mathcal{O}}^{-1} \vec{\sigma}$ we can easily check that the determined components of Σ_{ij} are of order $\Sigma_{ij} \sim \mu_B n k B / \omega_c^{(\text{ch})}$. This means that the contributions from Σ_{ij} are sufficiently small to be neglected in (4). However, in general we must also account for the contribution $\Sigma_{ij} = \Phi b_i b_j$ whose magnitude is unknown. Since the scalar field Φ cannot be determined if the three first terms of (5) are omitted, we must extend our model to solve the full case of Eq. (5). We will do so within a one-fluid model in the section III A, and it turns out that Φ becomes sufficiently large for this component of Σ_{ij} to significantly influence the solutions to (4), also within the MHD-regime. However, it also turns out that in order to get a large value of Φ , we must have a spin vector that is different from $\pm \mathbf{b}$. This is the normal case in a one-fluid theory, where the macroscopic spin results from averaging over all spin states. However, within a two-fluid MHD model (treating spin-up and down states as different species) without spin-velocity correlations Eq. (9) can be applied leading to $S_i = \pm b_i$, and the situation would then again be modified. Indeed, contracting Eq. (5) with $b_i b_j$ to compute the source terms for Φ , we find that all the source terms for Σ_{ij} vanishes if $S_i = \pm b_i$, as the fourth term in Eq. (5) becomes $b_i b_j P_{jk}(\partial S_i/\partial x_j)$, which is zero as $b_i(\partial S_i/\partial x_j) = \pm(1/2)\partial(b_i b_i)/\partial x_j$, whereas terms 7 and 8 together

become

$$b_i b_j \frac{\partial B_i}{\partial x_j} - b_i b_j S_i S_k \frac{\partial B_k}{\partial x_j} = 0$$

where $S_i = \pm b_i$ was used in the last step. This result provides the theoretical basis for adopting a two-fluid model of electrons in the MHD-regime and omitting spin-velocity correlations in (4), leading to $S_i = \pm b_i$. The division into two fluids leaves the rest of the basic equations structurally unaffected, but we now obtain two contributions such that the magnetization is calculated as $\mathbf{M} = \mu n_{\uparrow} \mathbf{s}_{\uparrow} + \mu n_{\downarrow} \mathbf{s}_{\downarrow}$ due to the difference in density perturbations of the two spin states. We will consider this in more detail within perturbation theory in our model problem below. The conclusions of this section is then confirmed, since the one-fluid models that keeps the spin-velocity correlations in Eq. (4) give indeed an identical expression for the magnetization as the two-fluid model with up and down spins $S_i = \pm b_i$. The allowance for independent density variations of the two species in the latter model provides the physical mechanism that reproduces the effects of spin-velocity correlations in the one-fluid model. It should however be stressed that this conclusion is limited to the MHD regime.

III. THREE WAVE INTERACTION - A MODEL PROBLEM

We will now consider a model problem with the purpose of testing our conclusions about the similarities between the one-fluid and two-fluid models outlined in the previous section. Specifically, we consider three wave interaction between two shear Alfvén waves (A, A') and one compressional Alfvén wave (MS); $MS \rightarrow A + A'$. Using three-wave interaction as a model problem has the advantage that an unphysical assumption (or an incorrect calculation) is likely to result in a broken Manley-Rowe symmetry [25], in which case one gets a clear indication that something needs to be revised.

The waves are assumed to be small perturbations on a homogeneous background, and we write $\mathbf{B} = B_0 \hat{\mathbf{z}} + \mathbf{B}_1$, $\rho = \rho_0 + \rho_1$, etc., but omit the index 1 on variables whose background values are zero. Furthermore, we omit index 1 whenever the cartesian components are specified for notational convenience, i.e. we write $\mathbf{B}_1 = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}$. Moreover, we assume that there is no drift so that $\mathbf{u}_0 = 0$ and also that there is no spin-velocity correlation in the background distribution, i.e. $\Sigma_0 = 0$. For simplicity we further assume that the temperature is sufficiently low such that the equilibrium pressure can be neglected, $P_0 = 0$ (i.e. that we have a low-beta plasma with the ion-acoustic velocity much smaller than the Alfvén velocity). Furthermore, assuming that $\mu_B B_0 / (k_B T) \ll 1$ we can make the approximation that the equilibrium spin up and spin down populations are equal so that $n_{0\uparrow} = n_{0\downarrow}$ in the two-fluid model which implies that the total zeroth order magnetization vanishes [26]. For consistency between the one-fluid and two-fluid models we should consequently pick $\mathbf{M}_0 = 0$ also in the latter case. The difference in the model equations between the one-fluid and two-fluid approach is then primarily that in the two-fluid model we have $\mathbf{S}_0 = \pm \hat{\mathbf{z}}$ for the two spin states (in which case we obtain a finite zero order magnetization if only one of the electron fluids are counted) whereas in the one-fluid model $\mathbf{M}_0 = 0$ and $\mathbf{S}_0 = 0$.

Next we make a harmonic decomposition $\partial_t \rightarrow -i\omega$ and $\partial_{\mathbf{x}} \rightarrow i\mathbf{k}$ for each wave, where the frequencies and wave vectors satisfy the conditions

$$\omega^{MS} = \omega^A + \omega^{A'} \quad (10)$$

$$\mathbf{k}^{MS} = \mathbf{k}^A + \mathbf{k}^{A'}. \quad (11)$$

with the index MS denoting the compressional Alfvén (or fast magnetosonic) wave, and the index A and A' denoting the shear Alfvén waves. The coordinate system is defined so that the z -direction points in the direction of the unperturbed magnetic field, $\mathbf{B} = B_0 \hat{\mathbf{z}}$, and for simplicity we assume all wave vectors to lie in the xz -plane.

Throughout the calculation we will use ω/ω_c and kC_A/ω_c as small expansion parameters (where $\omega_c = qB_0/m$ is the cyclotron frequency), in accordance with standard MHD theory. Here $C_A = (B_0^2/\mu_0 m_i n_0)^{1/2} = (B_0^2/\mu_0 \rho_0)^{1/2}$ is the Alfvén velocity, and ω and k represents any of the wave frequencies or wave vector components. We also note that $\omega_{cg} \simeq \omega_c$, where $\omega_{cg} = 2\mu_e B_0/\hbar$ is the spin precession frequency. Furthermore, $\omega_{cg} - \omega_c$ is of the same order as the ion-cyclotron frequency, and is therefore much larger than wave frequencies within the MHD regime [27]. We will therefore drop terms proportional to $(\omega_{cg} - \omega_c)^{-1}$ compared to ω^{-1} in our final results.

It should be pointed out that unlike the classical case, the pressure tensor P_{ij} does not necessarily vanish in the limit of zero temperature. However, we note that we need not be concerned about the contribution from the pressure term in this particular case. This is because P_{ij} vanishes linearly in the MHD limit and thereby enters as a cubic nonlinearity (which does not affect the three-wave interaction) in the evolution equation for Σ_{ij} . The pressure tensor, however, also gives a contribution in the MHD equation (6), but it turns out that this is a nonlinear contribution proportional to $(\omega_{cg} - \omega_c)^{-1}$ which is small compared to leading terms proportional to ω^{-1} . We may therefore neglect the contribution from P_{ij} altogether.

A. One-fluid calculation

We start by considering the one fluid model for which, as mentioned above, the unperturbed spin-density is zero, $\mathbf{S}_0 = 0$. Our first aim is to find the linear dispersion relation as well as the linear eigenvectors (polarizations) of the shear Alfvén wave and the compressional Alfvén wave. We note that in the MHD equation (6) we need an expression for the magnetization. We therefore begin by solving the spin-velocity evolution equation to find the Σ -tensor. Linearly, this is straightforward and we find

$$\Sigma_{ij} = in_0\mu \begin{pmatrix} -\frac{k_x B_y \omega_{cg}}{\omega_c^2 - \omega_{cg}^2} & -\frac{k_x B_x \omega_c}{(\omega_c^2 - \omega_{cg}^2)} & \frac{k_z B_y}{\omega_{cg}} \\ \frac{k_x B_x \omega_{cg}}{(\omega_c^2 - \omega_{cg}^2)} & -\frac{k_x B_y \omega_c}{\omega_c^2 - \omega_{cg}^2} & -\frac{k_z B_x}{\omega_{cg}} \\ 0 & -\frac{k_x B_z}{\omega_c} & i\frac{k_z B_z}{\omega} \end{pmatrix}. \quad (12)$$

As can be seen, the components in (12) have different magnitudes, but we keep all of them at this stage in the calculation. Next we use the linear Σ -tensor in the spin evolution equation (4) to find an expression for the linear spin \mathbf{S} and thereby the linear magnetization $\mathbf{M} = \mu n_0 \mathbf{S}$.

$$\mathbf{M} = \frac{n_0\mu^2}{m} \begin{pmatrix} \frac{k_x^2 B_x}{(\omega_c^2 - \omega_{cg}^2)} \\ \frac{k_x^2 B_y}{(\omega_c^2 - \omega_{cg}^2)} \\ -\frac{k_z^2 B_z}{\omega^2} \end{pmatrix} \quad (13)$$

Here we have dropped contributions to components of \mathbf{M} that are smaller by factors $|\omega_c^2 - \omega_{cg}^2|/\omega_{cg}^2$ and/or ω/ω_{cg} . Substituting (13) into (6), the linear dispersion relations are obtained from (6), (7) and (8). Similarly to the classical ideal MHD case, the modes decouple into the shear Alfvén wave described by

$$D_A(\omega, \mathbf{k}) \equiv \omega^2 - k_z^2 C_A^2 \left(1 - \frac{n_0\mu_0\mu^2}{m} \frac{k_x^2}{(\omega_c^2 - \omega_{cg}^2)} \right) = 0 \quad (14)$$

and the compressional Alfvén (fast magnetosonic) wave with the dispersion relation

$$D_{MS}(\omega, \mathbf{k}) \equiv \omega^2 - k_x^2 C_A^2 \left(1 + \frac{n_0\mu_0\mu^2}{m} \frac{k_z^2}{\omega^2} \right) - k_z^2 C_A^2 \left(1 - \frac{n_0\mu_0\mu^2}{m} \frac{k_x^2}{(\omega_c^2 - \omega_{cg}^2)} \right) = 0. \quad (15)$$

Note that the last terms in (14) and (15) are smaller than the first spin-modification in (15) as $\omega^2 \ll |\omega_c^2 - \omega_{cg}^2|$. Furthermore, the linear eigenvector components for the shear Alfvén wave are $u_x^A = u_z^A = 0$, $B_x^A = B_z^A = 0$, $\rho_1^A = 0$, and

$$B_y^A = -\frac{k_z^A B_0}{\omega^A} u_y^A \quad (16)$$

For the compressional Alfvén wave (index MS) we instead obtain $u_y^{MS} = u_z^{MS} = 0$, $B_y^{MS} = 0$, and

$$B_x^{MS} = -\frac{k_z^{MS} B_0}{\omega^{MS}} u_x^{MS} \quad (17)$$

$$B_z^{MS} = \frac{k_\perp^{MS} B_0}{\omega^{MS}} u_x^{MS} \quad (18)$$

$$\rho_1^{MS} = \rho_0 \frac{k_\perp^{MS}}{\omega^{MS}} u_x^{MS} \quad (19)$$

Next we aim to calculate the three wave coupling coefficients due to the quadratic nonlinearities. We have calculated the nonlinear contribution to the coupling coefficients including all terms proportional to $(\omega_{cg} - \omega_c)^{-1}$. However, our results show that these terms only give rise to small corrections to the leading terms proportional to ω^{-1} . Since the full analysis is rather tedious we will therefore only write out the leading terms in the NL contribution to the Σ -tensor as well as to the magnetization \mathbf{M} . Under the given approximations, keeping the resonant terms, we find the components with a nonzero nonlinear contribution to the Σ -tensor to be

$$\Sigma_{yz}^{A'} = -\frac{k_z B_x^{A'}}{\omega_{cg}} + \frac{2i\mu}{\hbar} \frac{k_z^{MS} B_z^{MS} B_y^{A*}}{\omega_{cg}\omega^{MS}} \quad (20)$$

and

$$\Sigma_{zy}^{A'} = -\frac{k_x B_z}{\omega_c} + \frac{iq}{m} \frac{k_z^{MS} B_z^{MS} B_y^{A*}}{\omega_c \omega^{MS}} \quad (21)$$

for the Alfvén wave. Here $*$ denotes complex conjugation. For the magnetosonic wave the component with a nonzero nonlinear contribution is

$$\Sigma_{33}^{MS} = i \frac{k_z B_z^{MS}}{\omega} + \frac{2i\mu}{\hbar} \frac{1}{\omega_{cg} \omega^{MS}} (k_z^A + k_z^{A'}) B_y^{A'} B_y^A \quad (22)$$

Solving the spin evolution equation with the sources from Σ given above, we find the a nonlinear contribution to the z -component of the magnetization

$$M_z^{MS} = -\frac{n_0 \mu^2}{m} \left(\frac{k_z^2 B_z^{MS}}{\omega^2} + \frac{2\mu}{\hbar} \frac{k_z^{MS} (k_z^A + k_z^{A'})}{\omega_{cg} \omega^{2(MS)}} B_y^A B_y^{A'} \right) \quad (23)$$

for the MS wave, and a nonlinear contribution to the y -component of the magnetization

$$M_y^{A'} = \frac{n_0 \mu^2}{m} \left(\frac{k_x^2 B_y}{(\omega_c^2 - \omega_{cg}^2)} - \frac{2\mu}{\hbar} \frac{k_z^{2(MS)}}{\omega_{cg} \omega^{2(MS)}} B_z^{MS} B_y^{A*} \right) \quad (24)$$

for the Alfvén wave. Now that we have expressed the magnetization in terms of the magnetic field, correct to second order in the amplitude, we may substitute these results into (6), and perform the rest of the calculations using (6), (7) and (8) as in standard MHD theory. Accounting for time dependent amplitudes with the substitutions $D_A(\omega, \mathbf{k}) \rightarrow [\partial D_A / \partial \omega] i \partial / \partial t$ and $D_{MS}(\omega, \mathbf{k}) \rightarrow [\partial D_{MS} / \partial \omega] i \partial / \partial t$, doing successive elimination keeping the velocity variables as the wave amplitudes, we find the following coupled equations for the different wave modes

$$\frac{\partial u_y^{A'}}{\partial t} = -i \frac{\omega^{2(A')}}{\partial D_{A'} / \partial \omega} C u_y^{A*} u_x^{MS} \quad (25)$$

and

$$\frac{\partial u_x^{MS}}{\partial t} = -i \frac{\omega^{2(MS)}}{\partial D_{MS} / \partial \omega} C u_y^A u_y^{A'} \quad (26)$$

with the coupling coefficient

$$C = \frac{k_x^{MS}}{\omega^{MS}} \left(1 + \frac{n_0 \mu_0 \mu^2}{m} \frac{k_z^{2(MS)}}{\omega^{2MS}} \right). \quad (27)$$

Due to the symmetry between the two shear Alfvén waves, the equation for $\partial u_y^A / \partial t$ is obtained by exchanging A and A' in Eq. (25). The appearance of the common factor C in the three coupled equations is a reflection of the Manley-Rowe symmetry [25]. The first term of C is a purely classical contribution, that agrees with e.g. Refs. [28, 29] in the cold limit. For the spin contribution in Eq. (27) to be important as compared to the classical one, a rather dense plasma is required. By contrast, other MHD phenomena exists that require less extreme parameters for the electron spin to be important. Nevertheless, as will be discussed in the final section, the results derived here have a number of interesting theoretical consequences. It should be stressed that the contribution to the magnetization in this one-fluid model stems from the Σ -tensor. This is in contrast to the two-fluid model as we will see below.

B. Two-fluid calculation

We now consider the problem of three wave coupling using the two-fluid model. The spin is then determined from (4) with the contribution from Σ omitted, as described in section II, but we now have two species of electrons, which have the unperturbed spin $\mathbf{S}_{0\uparrow} = \hat{\mathbf{z}}$ and $\mathbf{S}_{0\downarrow} = -\hat{\mathbf{z}}$, respectively. The total magnetization is then written as

$$\mathbf{M} = \mu \left(n^{(\uparrow)} \mathbf{S}^{(\uparrow)} + n^{(\downarrow)} \mathbf{S}^{(\downarrow)} \right) \quad (28)$$

which gives $\mathbf{M}_0 = 0$ in agreement with the previous section, provided we let $n_{0\uparrow} = n_{0\downarrow} = n_0/2$ which will be used henceforth. Next we find the linear spin-vector to be

$$\mathbf{S}_1 = \begin{pmatrix} \frac{2\mu}{\hbar\omega_{cg}} S_0 B_x + \frac{\mu}{m} \frac{k_x^2 B_x}{\omega_c^2 - \omega_{cg}^2} \\ \frac{2\mu}{\hbar\omega_{cg}} S_0 B_y + \frac{\mu}{m} \frac{k_y^2 B_y}{\omega_c^2 - \omega_{cg}^2} \\ 0 \end{pmatrix}. \quad (29)$$

Note here that although the terms $\propto S_0$ in (29) are larger than the terms $\propto (\omega_c^2 - \omega_{cg}^2)^{-1}$, the former has opposite signs for the up- and down species, and hence give no contribution to the linear magnetization. It turns out that the terms in (29) $\propto (\omega_c^2 - \omega_{cg}^2)^{-1}$ are needed to get agreement with the linear magnetization obtained with the one-fluid model (13). However, it can be noted that these terms have been dropped in the expression for the coupling coefficient (27) where, in the end, only the leading term is kept. Next we need to find an expression for the fluid densities of the electron spin fluids. From the continuity equation (1) we have

$$n^{(s)} = n_0^{(s)} + n_0^{(s)} \frac{\mathbf{k} \cdot \mathbf{v}^{(s)}}{\omega} + n^{(s)\text{NL}} \quad (30)$$

where $n^{(s)\text{NL}}$ is a non-linear contribution that can be shown not to contribute to the magnetization after summation of the spin states $s = \uparrow, \downarrow$. An expression for the electron velocities is obtained by solving the fluid momentum equation (2) together with $-\partial_t \mathbf{B} = \nabla \times \mathbf{E}$. To close this system we may in general need to use a full fluid description. However, within the MHD regime and for this specific problem, it suffices to determine the magnetization. For our case with $n_{0\uparrow} = n_{0\downarrow} = n_0/2$ several terms vanish in Eq. (28) after the summation over up and down species. In MHD we make the approximation that we may write the electric field as $\mathbf{E} = -\mathbf{v}^{(i)} \times \mathbf{B} \simeq -\mathbf{u} \times \mathbf{B}$. This allows us to again make use of Eqs. (16)-(19). Under these assumptions we note that E_z vanishes linearly, and also that E_z^{NL} is only proportional to quadratic combinations of B -field components and will therefore not contribute to the magnetization. Thus E_z may therefore be set to zero from now on. Under these assumptions, it is easy to show that the fluid velocities may be written as

$$v_x = \frac{q}{m} \frac{\omega}{\omega_c} \frac{B_z}{k_x} + \frac{q}{m} \frac{1}{\omega_c} (-v_x B_z + v_z B_x) \quad (31a)$$

$$v_y = -\frac{q}{m} \frac{\omega}{\omega_c} \frac{B_y}{k_z} + i \frac{\mu}{m\omega_c} k_x B_z S_0 - \frac{q}{m} \frac{1}{\omega_c} (v_y B_z - v_z B_y) - i \frac{\mu}{m\omega_c} k_x (B_x S_x - B_y S_y) \quad (31b)$$

$$v_z = -\frac{\mu}{m\omega} k_z B_z S_0 + i \frac{q}{m} \frac{1}{\omega} (v_x B_y - v_y B_x) - \frac{\mu}{m\omega} k_z (B_x S_x - B_y S_y) \quad (31c)$$

From Eqs. (28)–(31) we find the linear magnetization, and it agrees identically with the expression obtained from the one-fluid model. Furthermore, an extended analysis gives agreement for the NL terms of the magnetization as well. Consequently, the coupling coefficients remain the same regardless if the one-fluid or two-fluid model are used to determine the magnetization. This corroborates the usefulness of the two fluid model in the MHD regime.

IV. DISCUSSION

In the present paper we have studied a recently presented fluid model accounting for the electron spin [16], and adopted it to the MHD regime. The main feature of the original model is that in addition to basic spin effects as the magnetic dipole force, spin precession, and the magnetization current it incorporates spin-velocity correlations. The spin-velocity correlations have been shown to be important for a number of spin plasma phenomena [16, 24]. Introducing the approximations appropriate for the MHD regime, it turns out that essentially the ordinary MHD equations are recovered, but with a magnetization that needs to be determined. This can be done in different ways. Firstly from a single fluid model of electrons, that besides the basic spin precession contains spin-velocity correlations. Or, secondly, from a two-fluid model where spin-up and spin-down electrons constitute different species. A theoretical argument is presented in Section II B suggesting that these two models are equivalent in the MHD regime. However, the equivalence argument depends on certain assumptions which is difficult to justify rigorously, and thus practical tests of the equivalence is valuable. For this purpose we have evaluated the different models using a nonlinear three wave interaction as a test problem. Specifically, we have computed the coupling coefficients between two shear Alfvén waves and a compressional Alfvén wave. A classical as well as quantum mechanical (spin) contribution to the coupling coefficients are found, and the coupling coefficients are indeed identical in the two models. Furthermore, the coupling coefficients obey the Manley-Rowe symmetries [25]. The Manley-Rowe relations is a reflection of the

underlying Hamiltonian structure [30] of the model. The fact that the coefficients preserve these relations strongly suggests that the approximations made when deriving the models are sound, as otherwise it is highly likely that the Manly-Rowe symmetries would be broken.

Since we have put forward two somewhat different models in this paper, one may ask which one that is most easy to use. For the analytical calculations made here, the degree of complexity is found to be roughly the same. However, the two-fluid model has a great advantage in case one would like to do Particle-In-Cell (PIC) simulations. In that case, the only modification of a standard code would be to have two species of electrons, and add a force proportional to $\pm\mu_e\nabla B$ in the momentum equation, as well as to compute $\mathbf{M} = \mu_e\mathbf{b}(n_\uparrow - n_\downarrow)$ to find the contribution from the Magnetization current in Amperes law. As the variables for the density and magnetic field are monitored throughout the PIC-simulations anyway, no new equations and only little extra complexity is added to the general concept. Developing a PIC-scheme incorporating spin-velocity correlations, however, is a much more cumbersome project, as the evolution equation for this object is not present in present PIC-schemes, and also such equations are considerably more complex. Furthermore, it is not clear that it is at all possible to model spin-velocity correlations as a single-particle property, which makes the adaption of this model to the PIC-scheme questionable conceptually. Thus we conclude that the two-fluid model is valuable for the purpose of incorporating electron spin effects in PIC-simulations, although we stress that the applicability of such an approach will be limited to the MHD regime.

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